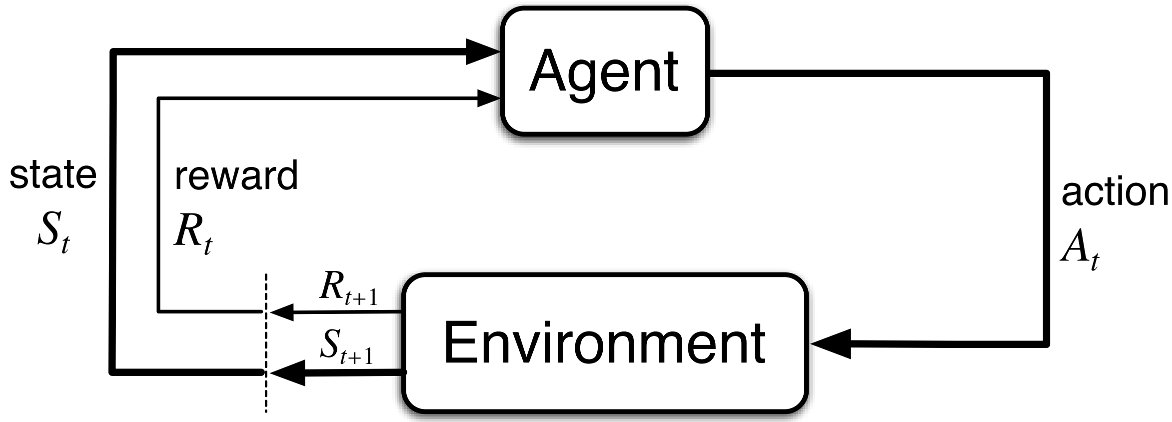
Chapter 3: The Reinforcement Learning Problem
(Markov Decision Processes, or MDPs)

Objectives of this chapter:

- present Markov decision processes—an idealized form of the AI problem for which we have precise theoretical results
- introduce key components of the mathematics: value functions and Bellman equations

The Agent-Environment Interface



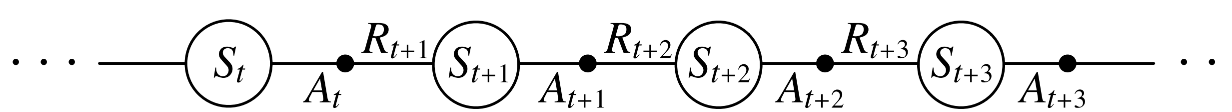
Agent and environment interact at discrete time steps: $t = 0, 1, 2, 3, \ldots$

Agent observes state at step t: $S_t \in \mathcal{S}$

produces action at step t : $A_t \in \mathcal{A}(S_t)$

gets resulting reward: $R_{t+1} \in \mathcal{R} \subset \mathbb{R}$

and resulting next state: $S_{t+1} \in \mathcal{S}^+$



Markov Decision Processes

- If a reinforcement learning task has the Markov Property, it is basically a **Markov Decision Process (MDP)**.
- If state and action sets are finite, it is a **finite MDP**.
- To define a finite MDP, you need to give:
  - **state and action sets**
  - one-step "dynamics"

$$p(s', r|s, a) = \mathbf{Pr}\{S_{t+1} = s', R_{t+1} = r \mid S_t = s, A_t = a\}$$

  - there is also:

$$p(s'|s, a) \doteq \Pr\{S_{t+1} = s' \mid S_t = s, A_t = a\} = \sum_{r \in \mathcal{R}} p(s', r|s, a)$$

$$r(s, a) \doteq \mathbb{E}[R_{t+1} \mid S_t = s, A_t = a] = \sum_{r \in \mathcal{R}} r \sum_{s' \in \mathcal{S}} p(s', r|s, a)$$

The Agent Learns a Policy

Policy at step t = π_t =

a mapping from states to action probabilities

$\pi_t(a \mid s) =$ probability that $A_t = a$ when $S_t = s$

Special case - *deterministic policies*:

$\pi_t(s) =$ the action taken with prob=1 when $S_t = s$

- Reinforcement learning methods specify how the agent changes its policy as a result of experience.
- Roughly, the agent's goal is to get as much reward as it can over the long run.

The Markov Property

- By "the state" at step *t*, the book means whatever information is available to the agent at step *t* about its environment.
- The state can include immediate "sensations," highly processed sensations, and structures built up over time from sequences of sensations.
- Ideally, a state should summarize past sensations so as to retain all "essential" information, i.e., it should have the **Markov Property**:

$$\mathbf{Pr}\{R_{t+1} = r, S_{t+1} = s' \mid S_0, A_0, R_1, \ldots, S_{t-1}, A_{t-1}, R_t, S_t, A_t\} =$$

$$p(s', r|s, a) = \mathbf{Pr}\{R_{t+1} = r, S_{t+1} = s' \mid S_t, A_t\}$$

- for all $s' \in \mathcal{S}^+, r \in \mathcal{R}$, and all histories $S_0,\ A_0,\ R_1,\ \ldots,\ S_{t-1},\ A_{t-1},\ R_t,\ S_t,\ A_t$.

The Meaning of Life
(goals, rewards, and returns)

Rewards and returns

- The objective in RL is to maximize long-term future reward
- That is, to choose A_t so as to maximize $R_{t+1}, R_{t+2}, R_{t+3}, \ldots$
- But what exactly should be maximized?
- The discounted *return* at time t:

the *discount rate*

$$G_t = R_{t+1} + \gamma R_{t+2} + \gamma^2 R_{t+3} + \gamma^3 R_{t+4} + \cdots \qquad \gamma \in [0, 1)$$

γ	Reward sequence	Return
0.5(or any)	1 0 0 0…	
0.5	0 0 2 0 0 0…	
0.9	0 0 2 0 0 0…	
0.5	-1 2 6 3 2 0 0 0…	

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$$G_4 = 0 \quad G_3 = 16 \quad G_2 = -4$$

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- And if $\gamma = 0.9$?

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- And if $\gamma = 0.9$?

$$G_1 = 130$$

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$$G_1 = 130 \quad G_0 =$$

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- And if $\gamma = 0.9$?

$$G_1 = 130 \quad G_0 = 118$$

4 value functions

	state values	action values
prediction	v_π	q_π
control	v_*	q_*

- All theoretical objects, mathematical ideals (expected values)

- Distinct from their estimates:

$$V_t(s) \qquad Q_t(s,a)$$

Values are *expected* returns

- The value of a state, given a policy:

$$v_\pi(s) = \mathbb{E}\{G_t \mid S_t = s, A_{t:\infty} \sim \pi\} \qquad v_\pi : \mathcal{S} \to \Re$$

- The value of a state-action pair, given a policy:

$$q_\pi(s,a) = \mathbb{E}\{G_t \mid S_t = s, A_t = a, A_{t+1:\infty} \sim \pi\} \qquad q_\pi : \mathcal{S} \times \mathcal{A} \to \Re$$

- The optimal value of a state:

$$v_*(s) = \max_\pi v_\pi(s) \qquad v_* : \mathcal{S} \to \Re$$

- The optimal value of a state-action pair:

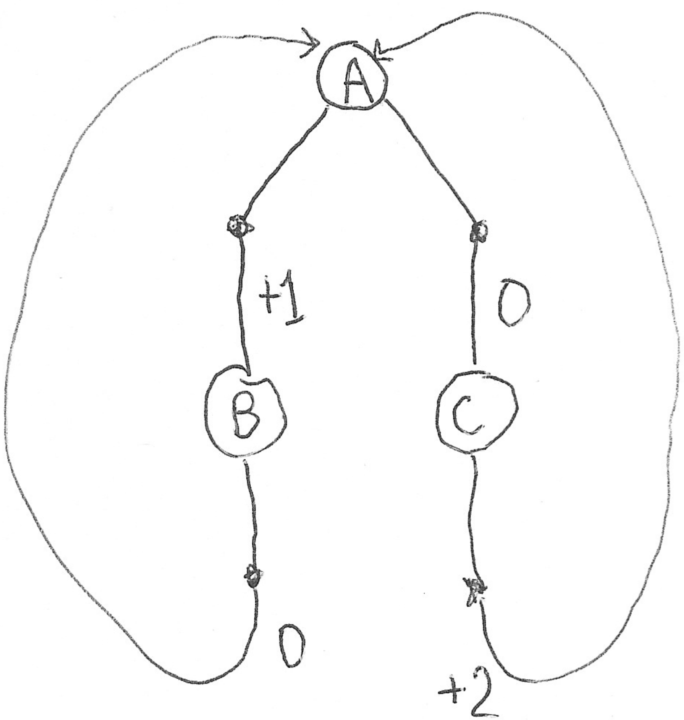
$$q_*(s,a) = \max_\pi q_\pi(s,a) \qquad q_* : \mathcal{S} \times \mathcal{A} \to \Re$$

- Optimal policy: π_* is an optimal policy if and only if

$$\pi_*(a|s) > 0 \text{ only where } q_*(s,a) = \max_b q_*(s,b) \qquad \forall s \in \mathcal{S}$$

  - in other words, π_* is optimal iff it is *greedy* wrt q_*

optimal policy example



What policy is optimal?

A: left

B: Right C: other

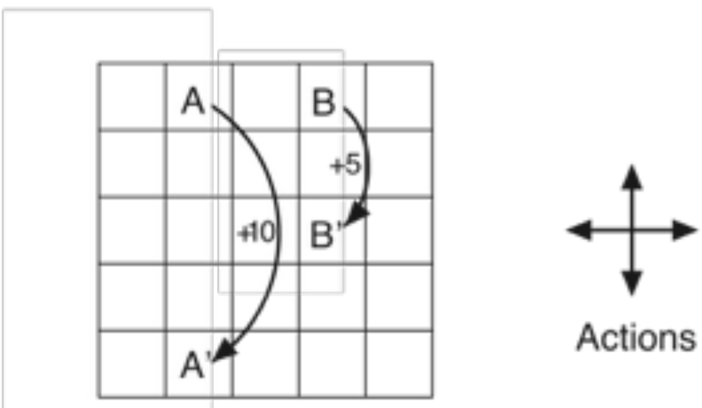
If $\gamma = 0$?

If $\gamma = .99$

If $\gamma = \frac{1}{2}$?

Gridworld

- Actions: `north`, `south`, `east`, `west`; deterministic.
- If would take agent off the grid: no move but reward = –1
- Other actions produce reward = 0, except actions that move agent out of special states A and B as shown.



(a)

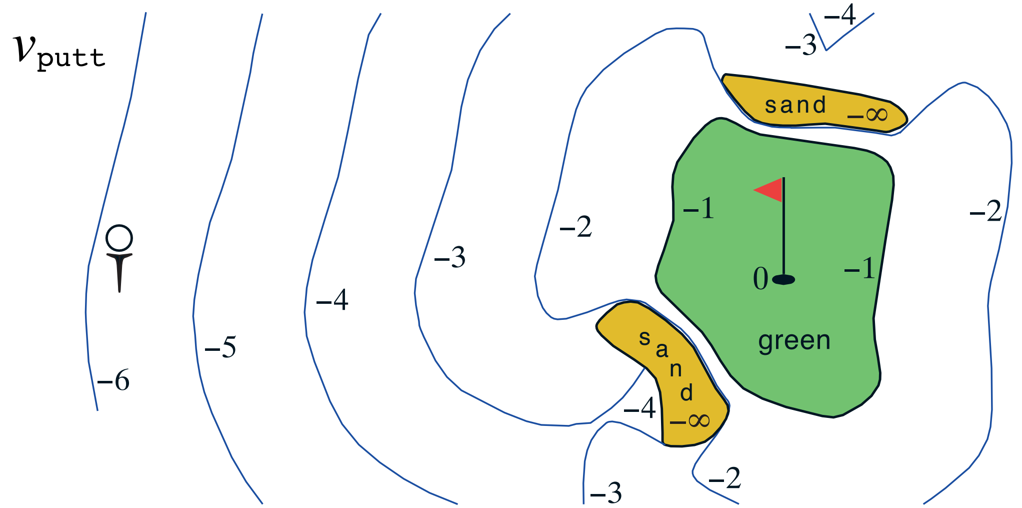
3.3	8.8	4.4	5.3	1.5
1.5	3.0	2.3	1.9	0.5
0.1	0.7	0.7	0.4	-0.4
-1.0	-0.4	-0.4	-0.6	-1.2
-1.9	-1.3	-1.2	-1.4	-2.0

(b)

State-value function for equiprobable random policy; $\gamma = 0.9$

Golf

- State is ball location
- Reward of –1 for each stroke until the ball is in the hole
- Value of a state?
- Actions:
  - `putt` (use putter)
  - `driver` (use driver)
- `putt` succeeds anywhere on the green



Optimal Value Functions

- For finite MDPs, policies can be **partially ordered**:

$$\pi \geq \pi' \quad \text{if and only if } v_\pi(s) \geq v_{\pi'}(s) \text{ for all } s \in \mathcal{S}$$

- There are always one or more policies that are better than or equal to all the others. These are the **optimal policies**. We denote them all π_*.

- Optimal policies share the same **optimal state-value function**:

$$v_*(s) = \max_\pi v_\pi(s) \quad \text{for all } s \in \mathcal{S}$$

- Optimal policies also share the same **optimal action-value function**:

$$q_*(s,a) = \max_\pi q_\pi(s,a) \text{ for all } s \in \mathcal{S} \text{ and } a \in \mathcal{A}$$

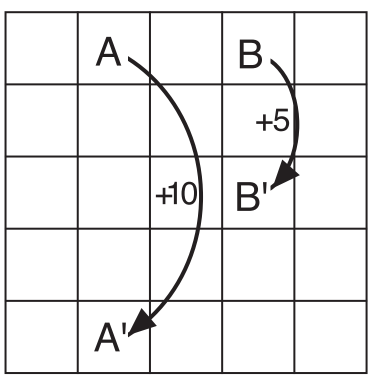
This is the expected return for taking action a in state s and thereafter following an optimal policy.

Why Optimal State-Value Functions are Useful

Any policy that is greedy with respect to v_* is an optimal policy.

Therefore, given v_*, one-step-ahead search produces the long-term optimal actions.

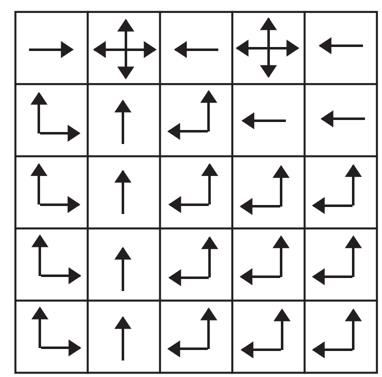
E.g., back to the gridworld:



a) gridworld

22.0	24.4	22.0	19.4	17.5
19.8	22.0	19.8	17.8	16.0
17.8	19.8	17.8	16.0	14.4
16.0	17.8	16.0	14.4	13.0
14.4	16.0	14.4	13.0	11.7

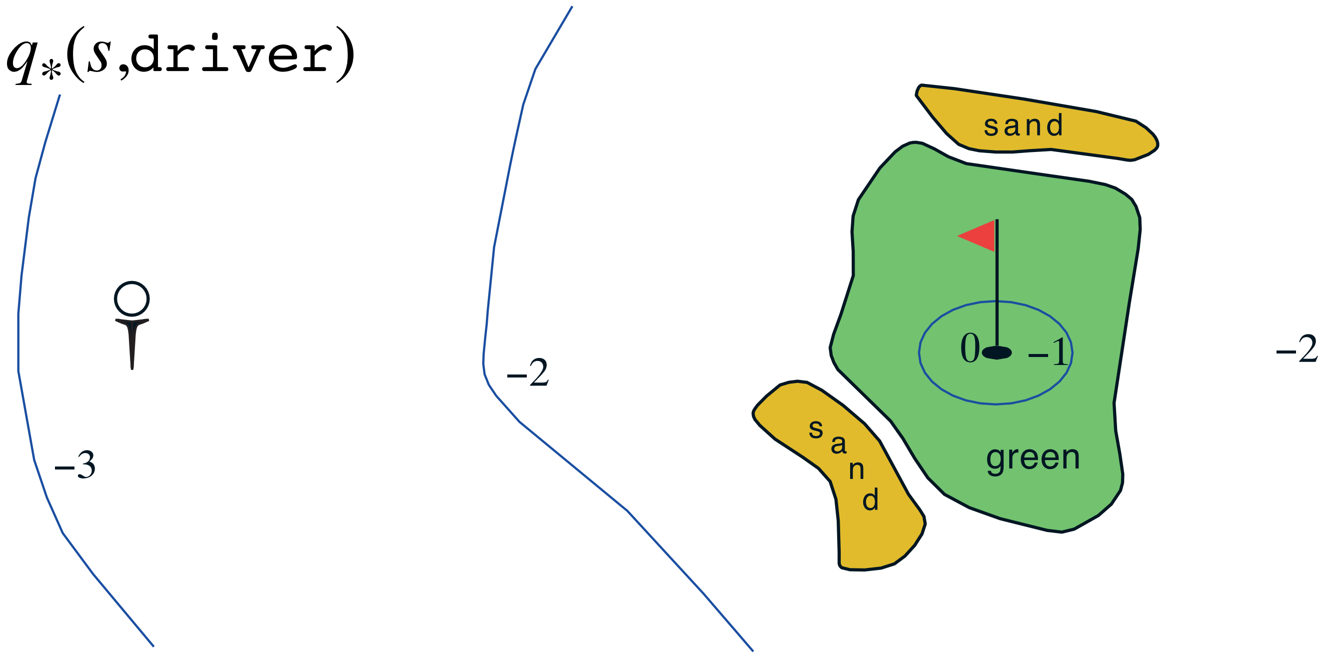
b) v_*



c) π_*

Optimal Value Function for Golf

- We can hit the ball farther with `driver` than with `putter`, but with less accuracy
- q_* (s,`driver`) gives the value or using `driver` first, then using whichever actions are best



What About Optimal Action-Value Functions?

Given q_*, the agent does not even have to do a one-step-ahead search:

$$\boxed{\pi_*(s) = \arg\max_a q_*(s,a)}$$

Value Functions
x 4

Bellman Equations
x 4

Bellman Equation for a Policy π

The basic idea:

$$\begin{aligned} G_t &= R_{t+1} + \gamma R_{t+2} + \gamma^2 R_{t+3} + \gamma^3 R_{t+4} + \cdots \\ &= R_{t+1} + \gamma \left(R_{t+2} + \gamma R_{t+3} + \gamma^2 R_{t+4} + \cdots \right) \\ &= R_{t+1} + \gamma G_{t+1} \end{aligned}$$

So:

$$\boxed{\begin{aligned} v_\pi(s) &= E_\pi \left\{ G_t \,\middle|\, S_t = s \right\} \\ &= E_\pi \left\{ R_{t+1} + \gamma v_\pi\left(S_{t+1}\right) \,\middle|\, S_t = s \right\} \end{aligned}}$$

Or, without the expectation operator:

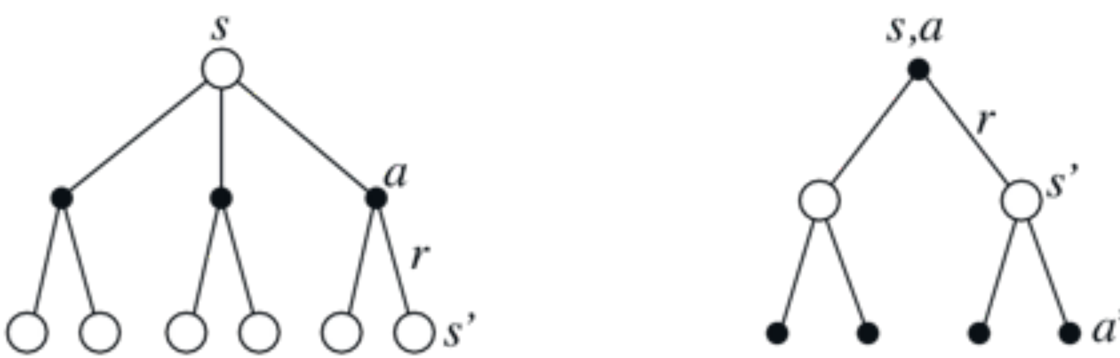
$$v_\pi(s) = \sum_a \pi(a|s) \sum_{s',r} p(s', r|s, a) \Big[r + \gamma v_\pi(s') \Big]$$

More on the Bellman Equation

$$v_\pi(s) = \sum_a \pi(a|s) \sum_{s',r} p(s', r|s, a)\Big[r + \gamma v_\pi(s')\Big]$$

This is a set of equations (in fact, linear), one for each state. The value function for π is its unique solution.

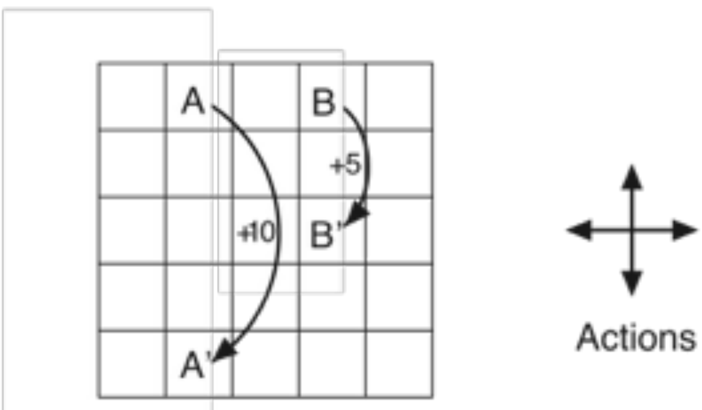
Backup diagrams:



for v_π for q_π

Gridworld

- Actions: `north`, `south`, `east`, `west`; deterministic.
- If would take agent off the grid: no move but reward = –1
- Other actions produce reward = 0, except actions that move agent out of special states A and B as shown.



(a)

3.3	8.8	4.4	5.3	1.5
1.5	3.0	2.3	1.9	0.5
0.1	0.7	0.7	0.4	-0.4
-1.0	-0.4	-0.4	-0.6	-1.2
-1.9	-1.3	-1.2	-1.4	-2.0

(b)

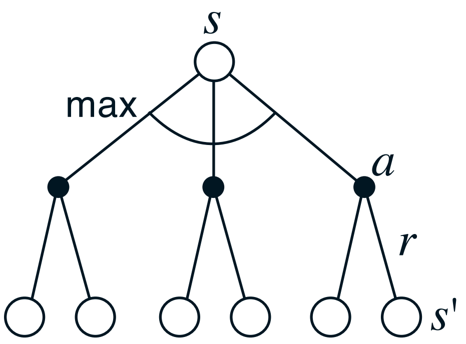
State-value function for equiprobable random policy; $\gamma = 0.9$

Bellman Optimality Equation for v_*

The value of a state under an optimal policy must equal the expected return for the best action from that state:

$$
\begin{aligned}
v_*(s) &= \max_a q_{\pi_*}(s,a) \\
&= \max_a \mathbb{E}[R_{t+1} + \gamma v_*(S_{t+1}) \mid S_t = s, A_t = a] \\
&= \max_a \sum_{s',r} p(s',r|s,a)\big[r + \gamma v_*(s')\big].
\end{aligned}
$$

The relevant backup diagram:

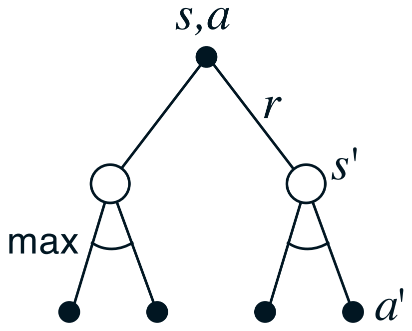


v_* is the unique solution of this system of nonlinear equations.

Bellman Optimality Equation for q_*

$$
\begin{aligned}
q_*(s,a) &= \mathbb{E}\Big[R_{t+1} + \gamma \max_{a'} q_*(S_{t+1}, a') \;\Big|\; S_t = s, A_t = a\Big] \\
&= \sum_{s',r} p(s', r | s, a)\Big[r + \gamma \max_{a'} q_*(s', a')\Big].
\end{aligned}
$$

The relevant backup diagram:



q_* is the unique solution of this system of nonlinear equations.

Why Optimal State-Value Functions are Useful

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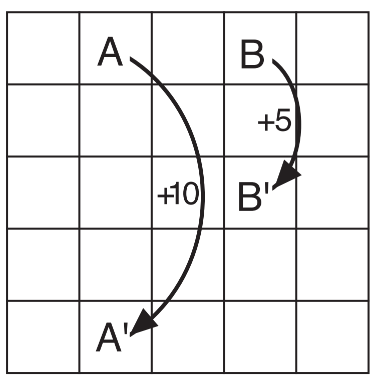
Therefore, given v_*, one-step-ahead search produces the long-term optimal actions.

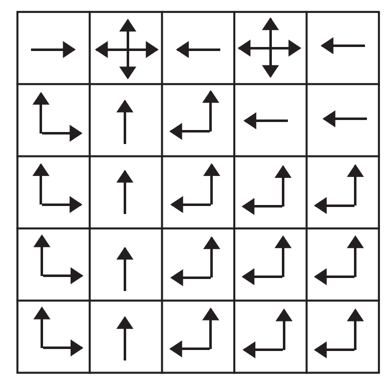
E.g., back to the gridworld:



a) gridworld

22.0	24.4	22.0	19.4	17.5
19.8	22.0	19.8	17.8	16.0
17.8	19.8	17.8	16.0	14.4
16.0	17.8	16.0	14.4	13.0
14.4	16.0	14.4	13.0	11.7

b) v_*



c) π_*

Solving the Bellman Optimality Equation

- Finding an optimal policy by solving the Bellman Optimality Equation requires the following:
  - accurate knowledge of environment dynamics;
  - we have enough space and time to do the computation;
  - the Markov Property.
- How much space and time do we need?
  - polynomial in number of states (via dynamic programming methods; Chapter 4),
  - BUT, number of states is often huge (e.g., backgammon has about 10^{20} states).
- We usually have to settle for approximations.
- Many RL methods can be understood as approximately solving the Bellman Optimality Equation.

Summary

- Agent-environment interaction
  - States
  - Actions
  - Rewards
- Policy: stochastic rule for selecting actions
- Return: the function of future rewards agent tries to maximize
- Episodic and continuing tasks
- Markov Property
- Markov Decision Process
  - Transition probabilities
  - Expected rewards
- Value functions
  - State-value function for a policy
  - Action-value function for a policy
  - Optimal state-value function
  - Optimal action-value function
- Optimal value functions
- Optimal policies
- Bellman Equations
- The need for approximation